Problem Set 8, Math 350, Fall 2017

(1) (a) Two square matrices $A, B$ of the same size are said to commute if $AB = BA$. Show that if $A, B$ commute then $A + B, B$ commute as well.

(b) Let $A, B$ be commuting matrices. Show that if $v$ is an eigenvector of $A$, then $Bv$ is also an eigenvector with the same eigenvalue.

(c) Show that if $A, B$ commute then any eigenvalue for $A + B$ is the sum of an eigenvalue for $A$ and an eigenvalue for $B$. (Hint: If $E_\lambda(A + B)$ is an eigenspace for $A + B$, then by parts (a) and (b) we have that $BE_\lambda(A + B) \subseteq E_\lambda(A + B)$. Let $v$ be an eigenvector for $B$ in $E_\lambda(A + B)$ .... )

(d) The discrete Laplacian (or discrete second derivative) is the operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by $(Lv)_k = (v_{k+1} - v_k) - (v_k - v_{k-1})$ where the indices are taken mod $n$ (so $v_{n+1} := v_1$). Find the matrix of $T$ with respect to the standard basis. (Your answer may use the ... notation. )

(e) Find the eigenvalues of the discrete Laplacian from part (d). (Hint: Write the matrix for $T$ as a sum of commuting matrices whose eigenvalues are easy to find and use part (c).)

(2) (a) A square matrix $A$ is stochastic if its entries are non-negative and the columns sum to 1. Show that the product of two stochastic matrices of the same size is stochastic.

(b) Show that for any stochastic $n \times n$ matrix $A$ and standard basis vector $e_j$, the entries of $A^k e_j$ are between 0 and 1. Use this to show that for any $n$ vector $v = [v_1 \ v_2 \ldots \ v_n]$ each entry of $A^k v$ is between $-\left(|v_1| + \ldots + |v_n|\right)$ and $+\left(|v_1| + \ldots + |v_n|\right)$.

(c) Using (b) show that the eigenvalues $\lambda$ of a stochastic matrix satisfy $|\lambda| \leq 1$. 